6 The classifying space of a topological groupoid

The *nerve* of a small category C is the simplicial set NC such that $N_n C$ is the set of functors

$$\{0, 1, \dots, n\} \to C,$$

where $\{0, 1, \dots, n\}$ is regarded as the category of pairs (i, j) where $i \leqslant j$ and composition is $(i, j)(j, k) = (i, k)$. This definition of nerve is due to Grothendieck, who also characterised simplicial sets of the form NC. If G is a groupoid, then G is also a category, and so its nerve NG is defined. (In fact NG has more structure, namely it is a 'simplicial T-complex of rank 1', as shown by Dakin, [46]. See also Ashley, [2].)

The geometric realisation $|NG|$ of the nerve of the groupoid G is called the *classifying space* BG of the groupoid G. It is a CW-complex, with one vertex for each element of $\mathrm{Ob}(G)$, one component for each component of G, and the fundamental group $\pi_1(BG, x), x \in \mathrm{Ob}(G)$ is isomorphic to the vertex group $G(x)$. Further, $\pi_i(BG, x) = 0, i > 1$. It is well-known that if X is any CW-complex then there'is a natural bijection

$$[X, BG] \cong [\pi_1 X, G]$$

between the set of (free) homotopy classes of maps $X \!-\!\!\to BG$ and the conjugacy classes of homomorphisms of groupoids $\pi_1 X \to G$.

This formula allows for a neat proof of a result of Gottlieb [67]. Let Y be a finite CW-complex, and let $(BG)^Y$ denote the space of (unpointed) maps $Y \to BG$. Then for any CW-complex X there is a sequence of natural bijections

$$\begin{aligned}
[X, (BG)^Y] &\cong [X \times Y, BG] \\
&\cong [\pi_1(X \times Y), G] \\
&\cong [\pi_1 X \times \pi_1 Y, G] \\
&\cong [\pi_1 X, \mathsf{HOM}(\pi_1 Y, G)] \\
&\cong [X, B(\mathsf{HOM}(\pi_1 Y, G))].
\end{aligned}$$

It follows that $(BG)^Y$ is of the homotopy type of $B(\mathsf{HOM}(\pi_1 Y, G))$. Note that if Y is connected, G is a group and $f : Y \to BG$ is a pointed map, then the vertex group of $\mathsf{HOM}(\pi_1 Y, G)$ at f_* is the centraliser of $f_*(\pi_1(Y))$ in G, which is the result of [67].

If G is a topological groupoid, then its nerve NG becomes a simplicial space. The realisation $BG = |NG|$ is still defined, but is no more a CW-complex [136].

The applications of this classifying space are legion. In the case G is a topological group, BG classifies principal bundles with group G. We mention some uses of the groupoid cases. When G is the groupoid of germs arising from a pseudo-group Γ, BG then classifies Γ–structures (see [66, 70, 96]). Also, the cohomology of $B\Gamma$ gives rise to characteristic classes for foliations [66].

If $G = X \rtimes H$, the semi-direct product topological groupoid arising from an action of the topological group H on the topological space X, then BG is also known as the homotopy limit of the action [140]. It is known that BG is of the homotopy type of the space $X \times_H PH$, where $PH \!-\!\!\to BH$ is a universal principal H-bundle. The *equivariant cohomology* of the H-space X is defined to be $H^*(X \times_H PH)$ [154], and is thus simply $H^*(BG)$ (compare [136, 153, 158]).